UPTEC 8450 R

MAY 1984

AN INPUT ESTIMATION APPROACH TO DIFFERENTIATION OF NOISY DATA

ANDERS AHLÉN

TEKNIKUM

INSTITUTE OF TECHNOLOGY

UPPSALA UNIVERSITY

ISSN 0346-8887

Uppsala Universitet
TEKNISKA HÖGSKOLAN
TEKNIKUM
INSTITUTIONEN FÖR TEKNOLOGI
Uppsala University
INSTITUTE OF TECHNOLOGY

UPTEC 8450 R

Date of issue
May 1984

Project name

Sponsoring organization

Author(s)

Ahlén, Anders

Titel (swedish)

Titel (english)

AN INPUT ESTIMATION APPROACH TO DIFFERENTIATION OF NOISY DATA

Abstract

An approach for estimating the derivative of a noisy signal is proposed and analysed. The idea of the method is to formulate the problem as estimation of the input to an n:th order integrator. Hence we have an input estimation problem. First the parameter of a suitable input model to the system are estimated using a prediction error method. Then the system is set up into state space form where the input is chosen to be one of the state variables. Standard fix point smoothing is applied to get an optimal estimate of the input. The method is applied to simulated data as well as real measurement data for illustration.

Keywords / indexing

Differentiation, Input estimation, Identification

Security

Language
English

Supplementary bibliographical information

Pages
50

Classification

ISSN 0346-8887

ISBN

TEKNIKUM
Box 534
751 21 Uppsala

INSTITUTE OF TECHNOLOGY
Box 534
S-751 21 Uppsala, Sweden

Gatuadress
Villavägen 4
Uppsala

Telefon
018 - 10 04 70

Telex
76143
UPTEC-S

AN INPUT ESTIMATION APPROACH TO DIFFERENTIATION OF NOISY DATA

ANDERS AHLEN
Automatic Control and Systems Analysis Group,
Department of Technology, Uppsala University,
P.O. Box 534, S-751 21 UPPSALA, Sweden.

ABSTRACT

An approach for estimating the derivative of a noisy signal is proposed and analysed. The idea of the method is to formulate the problem as estimation of the input to an n:th order integrator. Hence we have an input estimation problem. First the parameters of a suitable input model to the system are estimated using a prediction error method. Then the system is set up into state space form where the input is chosen to be one of the state variables. Standard fix point smoothing is applied to get an optimal estimate of the input. The method is applied to simulated data as well as real measurement data for illustration.

TABLE OF CONTENTS

1. INTRODUCTION

The problem of estimating the derivative of a noise corrupted signal has attracted a lot of attention in various fields. In biomechanics for instance, it is of great interest to compute different forces and torques from measurements of positions. We are then dealing with the difficult problem of estimating accelerations from noisy measurements of positions.

To overcome the difficulties in differentiating noisy measurements, a number of more or less sophisticated methods have been proposed. See for instance Rabiner and Steiglitz (1970), Anderssen and Bloomfield (1974), Wahba (1975), Gustafsson and Lanshammar (1977), Hatze (1981), Söderström (1980, 1982), and Usui and Amidror (1982).

In this report we will approach the differentiating problem by regarding it as a special case of the more general "input estimation problem". To estimate the derivative of a noise corrupted signal is then the same problem as to estimate the input of an integrator from noisy output measurements. In presence of noisy measurements, this problem is known to be ill-posed. Problems of this kind have been treated e.g. by Jakeman and Young (1978, 1981), Sandell et al (1981) and Candy and Zicker (1982).

We will investigate three different approaches of modelling the system and the input. In the first approach the input and the system are described by continuous-time models. The two other approaches are dealing with discrete-time models where the system is obtained either by constant or linear sampling.

First the parameters of an appropriate input model to the system are estimated using a prediction error method. Then the system is set up in a state space form where the input is chosen as one of the states. Optimal time invariant fix-point smoothing is then applied to estimate the input. The idea is not new. Similar ideas have been used e.g. by Young and Jakeman (1981), Candy and Zicker (1982). However, they use a fix input model determined with some rules of thumb. In this paper we determine the most suitable model using identification techniques. The most suitable model is assumed to be the one which minimizes the prediction errors of the output. Such an approach has also been used by Tugnait (1983) in connection with spectral estimation.

2. DIFFERENTIATION AS AN INPUT ESTIMATION PROBLEM

2.1 A CONTINUOUS-TIME APPROACH

Assume that the n:th order derivative $s^{(n)}(t)$, of a signal s(t) is to be determined. Call this derivative u(t). We shall in the following refer to u(t) as the derivative or the input. The signal s(t) can then be regarded as the output of an n:th order integrator, driven by u(t), see figure 2.1.

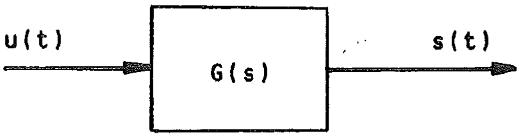


Figure 2.1 $G(s) = 1/s^n$

The problem of finding the derivative u(t) has now been formulated as the problem of determining the input to a given system from measurements of the output. Assume for a while that G(s) is more general than an n:th order integrator i.e. given by

$$G(s) = \frac{B(s)}{A(s)}$$

where

$$B(s) = b_{m-p}s^p+\ldots+b_m \qquad (2.1)$$

$$A(s) = s^m+a_1s^{m-1}+\ldots+a_m$$

We shall assume that the signal s(t) can be described as a continuous-time stationary stochastic process, generated by a continuous-time white noise e(t) through a filter H(s). The poles of H(s) are assumed to be inside the left half plane and the zeros in the left half plane or on the imaginary axis, cf Åström 1970. Then we can write

$$H(s) = \frac{\bar{C}(s)}{\bar{D}(s)}$$

where

$$\bar{C}(s) = \bar{c}_{r-q}s^q+\ldots+\bar{c}_r \qquad (2.2)$$

$$\bar{D}(s) = s^r+\bar{d}_1s^{r-1}+\ldots+\bar{d}_r$$

The input u(t) can then be described by

$$u(t) = \frac{C(s)}{D(s)}\, e(t) \qquad (2.3a)$$

where C(s) and D(s) must contain A(s) and B(s) respectively as factors i.e.

$$C(s) = \bar{C}(s)A(s)$$

$$D(s) = \bar{D}(s)B(s) \qquad (2.3b)$$

$$C(s) = c_{r+p-q-m}s^{q+m} + \ldots + c_{r+p}$$

$$D(s) = s^{r+p} + d_1 s^{r+p-1} + \ldots + d_{r+p}$$

If the input u(t) should be stationary we then have to require that B(s) has all zeros inside the left half plane and A(s) all zeros inside the left half plane or on the imaginary axis.

The above relations are depicted in figure 2.2.

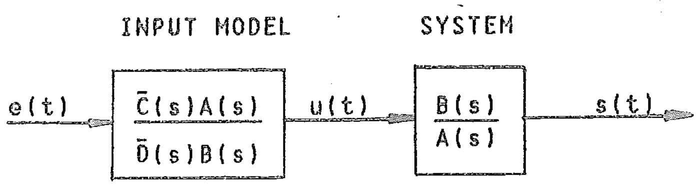


Figure 2.2 The input u(t) is described by a continuous-time stochastic process generated by a white noise through a filter that is stable and minimum phase.

To ensure the stationary properties of u(t) and then also on s(t) we have to require that the degree of C(s) will be at least one unit lower than the degree of D(s), i.e.

$$0 \leq q < r+p-m \qquad (2.4)$$

This give us a condition on how the signal has to be described if it should be possible to determine the input u(t).

We can represent (2.1) and (2.3) in state space form. Formally we can write, cf Åström (1970),

$$\frac{dx}{dt} = Ax(t) + B\frac{d\tilde{e}}{dt} \qquad (2.5)$$

$$s(t) = Cx(t)$$

where $d\tilde{e}/dt = e(t)$ and $\tilde{e}(t)$ is a Wiener process with unit variance parameter. Thus we have for (2.1) using observable form

$$\frac{d\bar{x}}{dt} = \begin{bmatrix} -a_1 & 1 & & & \mathbf{0} \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ \cdot & & & & \\ \cdot & \mathbf{0} & & & \cdot 1 \\ -a_m & & & & 0 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{m-p} \\ \vdots \\ \cdot \\ b_m \end{bmatrix} u(t) \qquad (2.6a)$$

$$s(t) = (10\ldots.0)\bar{x}(t)$$

and for (2.3)

$$\frac{dz}{dt} = \begin{bmatrix} -d_1 & 1 & & & & \mathbf{0} \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ \cdot & \mathbf{0} & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & 1 \\ -d_{r+p} & & & & & 0 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ c_{r+p-q-m} \\ \cdot \\ \cdot \\ \cdot \\ c_{r+p} \end{bmatrix} e(t) \qquad (2.6b)$$

$$u(t) = (10\ldots.0)z(t)$$

Combining (2.6a) and (2.6b) into one system on the form (2.5) with the augmented state vector $x(t) = (\bar{x}(t)^T\ z(t)^T)^T$ we obtain

$$\frac{dx}{dt} = \left[\begin{array}{ccccc|cccc} -a_1 & 1 & & & & 0 & & & \\ \cdot & & \cdot & & & \vdots & & \mathbf{0} & \\ \cdot & & & \cdot & & 0 & & & \\ \cdot & & & & \cdot & b_{m-p} & & & \\ \cdot & & & & 1 & \vdots & & & \\ -a_m & & & & 0 & b_m & & & \\ \hline & & & & & -d_1 & 1 & & \\ & & & & & \cdot & \cdot & \mathbf{0} & \\ & & & & & \cdot & & \cdot & \\ & & \mathbf{0} & & & \cdot & & & \cdot \\ & & & & & \cdot & \mathbf{0} & & \cdot \\ & & & & & \cdot & & & \cdot \\ & & & & & \cdot & & & 1 \\ & & & & & -d_{r+p} & & & 0 \end{array}\right] x(t) + \left[\begin{array}{c} \\ \mathbf{0} \\ \\ \hline \left.\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}\right\} \Delta-1 \\ c_{r+p-q-m} \\ \cdot \\ \cdot \\ \cdot \\ c_{r+p} \end{array}\right] e(t)$$

(2.7)

$$s(t) = (10..0\,\vdots\,0\ldots.0)x(t)$$

$u(t) = (0...0 \vdots 10...0)\ x(t)$

$\Delta = \deg(D) - \deg(C)$

We will assume that the measurements are given in discrete points. The sampled representation of (2.7) will then be, cf Åström (1970),

$$x(t+T) = Fx(t) + v(t)$$

$$y(t) = Cx(t) + w(t) \qquad (2.8)$$

$$u(t) = Hx(t)$$

where

$$F = e^{AT} \text{ ; } T = \text{sampling interval} \qquad (2.9)$$

C and H are defined in (2.7)

and $\{v(t)\}$ is an n-dimensional uncorrelated sequence with zero-mean and covariance matrix

$$R_d \triangleq \int_0^T e^{A\tau}BB^Te^{A\tau^T}d\tau \qquad (2.10)$$

We have in (2.8) assumed the disturbance effects on (2.7) to be white and additive to the output measurements. They are described by w(t) which also is assumed to be uncorrelated with v(t).

It is now straight-forward to estimate the input u(t) as

$$\hat{u}(t) = H\hat{x}(t) \qquad (2.11)$$

This problem will be further discussed in section 2.5.

If we return to the more specialized differentiation problem, G(s) in (2.1) will be chosen as

$$B(s) = 1$$

$$A(s) = s^n \qquad (2.12)$$

and

$$0 \leq q < r+p-m = r-n$$

With q = r-n-1 (2.7) will be

$$\frac{dx}{dt} = \left[\begin{array}{ccccc|ccccc} 0 & 1 & & & & 0 & & & & \\ \cdot & & \cdot & & & \cdot & & & & \\ \cdot & & & \cdot & & \cdot & & \mathbf{0} & & \\ \cdot & & & & \cdot & \cdot & & & & \\ \cdot & & & & 1 & \cdot & & & & \\ 0 & & & & 0 & 1 & & & & \\ \hline & & & & & -d_1 & 1 & & & \\ & & & & & \cdot & & \cdot & & \\ & & \mathbf{0} & & & \cdot & & & \cdot & \\ & & & & & \cdot & & & & \cdot \\ & & & & & \cdot & & & & 1 \\ & & & & & -d_r & & & & 0 \end{array}\right] x(t) + \left[\begin{array}{c} \\ \mathbf{0} \\ \\ \hline c_{n+1} \\ \vdots \\ c_r \\ 0 \\ \vdots \\ 0 \end{array}\right] e(t) \qquad (2.13)$$

$s(t) = (10..0 \mid 0....0)x(t)$

$u(t) = (0...0 \mid 10...0)x(t)$

If we in (2.13) use controllable form instead, the input, or in this case the n:th derivative will be

$$s^{(n)}(t) = (0...0|c_{n+1}...c_r 0...0)x(t) =$$

$$= (c_{n+1}...c_r 0...0)z(t)$$

This is exactly the same expression for the n: th derivative as derived in Söderström (1980, 1982). From (2.12), (2.13) we then conclude that the signal s(t) is generated by continuous-time white noise through the filter

$$\frac{\bar{C}(s)}{\bar{D}(s)} = \frac{\bar{c}_{n+1}s^{r-n-1}+...+\bar{c}_r}{s^r+\bar{d}_1 s^{r-1}+...+\bar{d}_r} \qquad (2.14)$$

To get a practical algorithm we must solve the following two problems.

1) Determine an appropriate input-model, i.e. C(s), D(s) and the variance of the continuous-time white noise $\lambda_{e_c}^2$. If the variance of the measurement noise is unknown it is necessary to determine that too.

2) Estimate the input u(t) based on the model derived in 1) and the measurements y(t), in some optimal sense.

These problems will be discussed in sections 2.4 and 2.5 respectively.

2.2 A DISCRETE-TIME APPROACH OBTAINED BY CONSTANT SAMPLING

Since the measurements of the output will be available in discrete time, it could be more convenient to have a discrete-time model relating u(t) and s(t). Such a model requires some assumptions of the properties of u(t) between the sampling points. One form of approximation is to derive the discrete-time model by constant sampling i.e. to assume u(t) to be constant between the sampling points.

If G(s) is $1/s^n$ we then recieve the discrete-time system, see Åström et al (1984),

$$H(q^{-1}) = \frac{T^n}{n!} \frac{B_n(q^{-1})}{(1-q^{-1})^n}$$

where

$$B_n(q^{-1}) = b_1^n q^{-1} + b_2^n q^{-2} + \ldots + b_n^n q^{-n} \tag{2.15a}$$

and

$$b_k^n = \sum_{l=1}^{k} (-1)^{k-l} l^n \binom{n+1}{k-l} \quad k=1\ldots n \ ; \ T = \text{the sampling interval}$$

For k = 1, 2 this will be

$$H_1(q^{-1}) = \frac{Tq^{-1}}{(1-q^{-1})}$$

(2.15b)

$$H_2(q^{-1}) = \frac{T^2/2!\ (q^{-1}+q^{-2})}{(1-q^{-1})^2}$$

The estimate of the derivative u(t) should then be thought of as a piece-wise constant signal. It can be argued that the input u(t) or specially the derivative, will in reality not be a piece-wise constant signal. However, there will not be any algorithmic problem letting u(t) be a piece-wise linear function nor to use other discrete approximations, e.g. Euler approximations, as well. We will return to this aspect in section 2.3. Due to its simplicity the approximation (2.15) will be discussed and used.

The discrete-time approximation described above for an n:th order integrator has worked well in practice, cf section 4, especially if the sampling interval is not chosen too large.

Assume now further that the input u(t) can be described by an m:th order ARMA-process according to figure 2.3

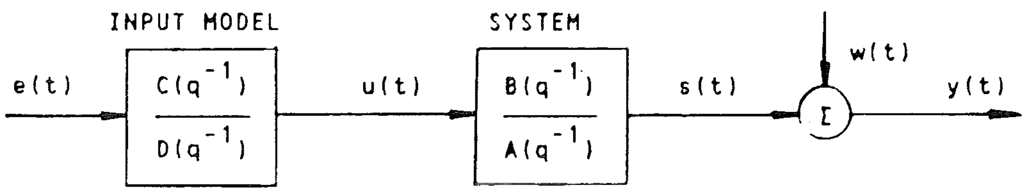


Figure 2.3 The input is described by an m:th order ARMA process. The measurement noise w(t) is assumed to be uncorrelated with the driving noise e(t).

Then the noisy signal y(t) can be described by

$$y(t) = \frac{C(q^{-1})B(q^{-1})}{D(q^{-1})A(q^{-1})} e(t) + w(t) \qquad (2.16a)$$

where A, B, C, and D are polynomials in the backward shift operator q^{-1} i.e.

$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_n q^{-n}$$

$$B(q^{-1}) = b_1 q^{-1} + \ldots + b_n q^{-n}$$

$$(2.16b)$$

$$C(q^{-1}) = 1 + c_1 q^{-1} + \ldots + c_m q^{-m}$$

$$D(q^{-1}) = 1 + d_1 q^{-1} + \ldots + d_m q^{-m}$$

The noise sequences {e(t)} and {w(t)} are assumed to be uncorrelated, with zero mean, and variances

$$Ee(t)^2 = \lambda_e^2$$

$$Ew(t)^2 = \lambda_w^2 \qquad (2.16c)$$

In (2.16a) we have used a more general notation for the system. If the system is described by (2.15) then u(t) will be an approximation of the derivative of s(t).

Since our aim is to estimate the input u(t) in some way it is natural to formulate (2.16a) in the same way as in (2.7), i.e. in state space form choosing u(t) as one of the state variables.

Using observer form as in (2.6) s(t) can be described by

$$\bar{x}(t+1) = \begin{bmatrix} -a_1 & 1 & & 0 \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & 0 & 1 \\ -a_n & & & 0 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix} u(t) \qquad (2.17)$$

$$s(t) = (10.....0)x(t)$$

and the input u(t) by

$$z(t+1) = \begin{bmatrix} -d_1 & 1 & & 0 \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & 0 & 1 \\ -d_m & & & 0 \\ 0 & \cdot\;\cdot\;\cdot & \cdot & 0 \end{bmatrix} z(t) + \begin{bmatrix} 1 \\ c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_m \end{bmatrix} e(t+1) \qquad (2.18)$$

$$u(t) = (10.....0)z(t)$$

Combining (2.17) and (2.18) into one state space model using the augmented vector $x(t) = (\bar{x}(t)^T z(t)^T)^T$ (2.16a) can be reformulated as in (2.13) i.e.

$$x(t+1) = \left[\begin{array}{cccc|cccc} -a_1 & 1 & & 0 & b_1 & & & \\ \vdots & & \ddots & & \vdots & & 0 & \\ \vdots & 0 & & 1 & \vdots & & & \\ -a_n & & & 0 & b_n & & & \\ \hline & & & & -d_1 & 1 & & 0 \\ & & & & \vdots & & \ddots & \\ & 0 & & & \vdots & & 0 & \cdot \\ & & & & \vdots & & & 1 \\ & & & & -d_m & & & 0 \\ 0 & \dots & & 0 & 0 & \dots & & 0 \end{array}\right] x(t) \;+\; \begin{bmatrix} 0 \\ \\ \dots \\ 1 \\ c_1 \\ \vdots \\ c_m \end{bmatrix} \bar{e}(t) \qquad (2.19)$$

$$y(t) = (10..0 \vdots 0...0)x(t) + w(t)$$

$$u(t) = (0...0 \vdots 10..0)x(t)$$

Standard optimal state estimation methods can now be applied to estimate the input u(t). This will be discussed in section 2.5.

In (2.19) we have assumed the measurement noise to be white. There will however not be any algorithmic problem to assume it coloured. We then simply have to add some extra states in (2.19) corresponding to the dynamics in the measurement noise. If no restrictions on the polynomials in (2.16b) are made this will probably lead to uniqueness problems and we will not discuss it further in this report.

Note that in (2.19) we have substituted e(t+1) by $\bar{e}(t)$ since the time index will not affect the properties of the input model nor the variance $\lambda_{\bar{e}}^2 = \lambda_e^2$ since {e(t)} is assumed to be a stationary process. To get a practical algorithm we must solve the two following problems.

1) Determine a pertinent model of the input i.e. the polynomials $C(q^{-1})$, $D(q^{-1})$ and the variance λ_e^2. If the variance of the measurement noise is unknown it is necessary to determine that too.

2) Estimate the input u(t), based on the model derived in 1), and the measurements y(t), in some optimal sense.

We will discuss these two problems in sections 2.4 and 2.5.

2.3 A DISCRETE-TIME APPROACH OBTAINED BY LINEAR SAMPLING

In section 2.2 we used constant sampling to obtain an approximation of the system G(s). We shall in this section use linear sampling instead. It means that we assume the input u(t) to vary linearly between the sampling points i.e.

$$u(t+\tau) = u(t) + \frac{\tau}{T}[(u(t+T) - u(t)] \quad ; 0\leq\tau\leq T \tag{2.20}$$

It is impossible to realize (2.20) in real time if u(t) should be used for control applications since u(t+T) is not known when t = τ. Some approximations are then in order. However, when we do not need to control, (2.20) is realizable, especially for off-line applications. When linear sampling is used, the discrete-time system obtained with (2.20) is described by the following lemma.

Lemma 2.1

Let the continuous-time system be described by

$$\dot{x} = Ax(t) + Bu(t) \tag{2.21a}$$

Assume that (2.21a) is sampled by letting the input u(t) be a linear function between the sampling points i.e

$$u(t+\tau) = u(t) + \frac{\tau}{T}[u(t+\tau) - u(t)] \quad ; 0 \leq \tau \leq T \tag{2.21b}$$

Then the discrete-time system is given by

$$x(t+T) = Fx(t) + G_1u(t) + G_2u(t+T) \tag{2.22a}$$

$$s(t) = Cx(t)$$

where

$$F = e^{AT}$$

$$G_1 = \int_0^T e^{A(T-\tau)}B(1-\frac{\tau}{T})d\tau$$

(2.22b)

$$G_2 = \int_0^T e^{A(T-\tau)}B\,\frac{\tau}{T}\,d\tau$$

In the special case when A is nonsingular we obtain

$$G_1 = \frac{1}{T} A^{-2}[e^{AT}(AT-I)+I]B$$

$$G_2 = \frac{1}{T} A^{-2}[e^{AT}-AT-I)]B \qquad (2.22c)$$

∎

Proof: See Appendix 1.

Equation (2.22c) is also given by Strmcnik and Bremsak (1979) In equation (2.22a) we need both u(t) and u(t+T) to compute x(t+T) i.e. we will have a direct term from the input to the output.

If especially the continuous-time system is given by $G(s) = 1/s^k$ we can describe the discrete-time system as follows:

Theorem 2.1

Let the linear sampled system be described by Lemma 2.1 and assume that $G(s) = 1/s^k$. Then the discrete-time system is given by

$$H_k(q) = \frac{T^k}{(k+1)!(q-1)^k} \sum_{j=1}^{k} B_j \binom{k+1}{j-1} (q-1)^{k-j}(q+k+1-j) \qquad k \geq 1 \quad (2.23)$$

$$B_j = \sum_{p=1}^{j-1} B_p \binom{j-1}{p-1} (q-1)^{j-1-p} \qquad j \geq 2$$

$$B_1 = 1 \qquad (2.24)$$

where T is the sampling interval and q is the forward shift operator i.e. $qy(t) = y(t+T)$. ∎

Proof: See Appendix 1.

For k=1,2 the discrete-time system will be, using the backward shiftoperator q^{-1}

$$H_1(q^{-1}) = \frac{T/2!(1+q^{-1})}{(1-q^{-1})}$$

$$H_2(q^{-1}) = \frac{T^2/3!\ (1+4q^{-1}+q^{-2})}{(1-q^{-1})^2} \qquad (2.25)$$

If we compare (2.25) with (2.15b) we can expect (2.25) to be a better approximation of G(s) than (2.15b) since (2.25) contains a direct term.

For a k: th order integrator let the discrete-time system be described by

$$H_k(q^{-1}) = \frac{\bar{b}_o+\bar{b}_1q^{-1}+\ldots+\bar{b}_mq^{-m}}{1+a_1q^{-1}+\ldots+a_mq^{-m}} \tag{2.26}$$

This can easily be incorporated into (2.19) by changing the C-matrix and the b-parameters in the A-matrix according to (2.27) i.e.

$$\begin{aligned} y(t) &= (10\ldots.0 \,\vdots\, \bar{b}_o0\ldots0)x(t) + w(t) \\ b_j &= \bar{b}_j-\bar{b}_oa_j \qquad j \geq 1 \end{aligned} \tag{2.27}$$

The discrete approximation described above for a k: th order integrator has worked well in practice. In section 4 we will compare the continuous-time representation with this approximation as well as the approximation derived in section 2.2.

The discussion in the previous section will apply even for this section with the modification (2.27). It should also be noted that in this case the derivative u(t) should be viewed as a piece-wise linear signal.

2.4 IDENTIFICATION OF THE INPUT MODEL

In this section we will discuss how to determine the input model parameters described by (2.3) and (2.16). We shall start with the discrete-time models and then do some modifications to suit the continuous-time model.

For the discrete-time models described by (2.19) let the parameters to be determined be denoted by the vector

$$\theta = [c_1 \ldots c_m \; d_1 \ldots d_m \; \lambda_e^2 \; \lambda_w^2]^T \tag{2.28}$$

One possibility is to use a priori knowledge for at least a crude determination of the parameter vector θ. Another approach, to be described in this subsection, is to estimate the parameter vector θ from measurements of the noisy output y(t). This idea has also been used by Tugnait (1983) for estimation of the power spectral density of a signal described by an ARMA-process.

We will describe how this can be done using a prediction error method, for details see Ljung (1976). For this purpose (2.19) is rewritten as

$$\begin{aligned} x(t+1,\theta) &= A(\theta)x(t,\theta) + v(t,\theta) \\ y(t,\theta) &= C(\theta)x(t,\theta) + w(t) \\ u(t,\theta) &= H(\theta)x(t,\theta) \end{aligned} \tag{2.29}$$

where $v(t,\theta) = B(\theta)\bar{e}(t)$

and θ is defined by (2.28).

The idea is to minimize the prediction errors of the output with respect to the parameter vector θ, i.e. to minimize

$$V_N(\theta) = \frac{1}{N}\sum_{t=1}^{N} ||y(t) - \hat{y}(t|t-1;\theta)||^2 \tag{2.30}$$

In order to find the optimal predictor $\hat{y}(t|t-1;\theta)$ consider the stationary Kalman filter of (2.29)

$$\begin{aligned} \hat{x}(t+1|t;\theta) &= A(\theta)\hat{x}(t|t-1;\theta) + K(\theta)(y(t) - C(\theta)\hat{x}(t|t-1;\theta)) \\ \hat{y}(t|t-1;\theta) &= C(\theta)\hat{x}(t|t-1;\theta) \end{aligned} \tag{2.31a}$$

where the gain vector $K(\theta)$ is determined by the Riccati equation

$$\begin{aligned} P &= APA^T + R_1 - APC^T(CPC^T + R_2)^{-1}CPA^T \\ K &= APC^T(CPC^T + R_2)^{-1} \end{aligned} \tag{2.31b}$$

and R_1, R_2 are given by

$$R_1 = Ev(t,\theta)v(t,\theta)^T = \lambda_e^2 B(\theta)B(\theta)^T = \lambda_e^2 \left[\begin{array}{c|cccc} 0 & & 0 & & \\ \hline & 1 & c_1 & \cdots & c_m \\ 0 & c_1 & \cdot & & \\ & \cdot & & \cdot & \\ & c_m & & & c_m^2 \end{array}\right] \tag{2.31c}$$

$$R_2 = Ew(t)^2 = \lambda_w^2$$

Note that the matrices A, C, P, K, R_1 and R_2 in (2.31b), (2.31c) depend on the parameter vector θ. The optimal prediction is then given by (2.31a).

The vector $\hat{\theta}$ that minimizes the criterion (2.30) will give an estimate of the "filter" that is believed to generate the input u(t). It is desirable that $\hat{\theta}$ is such that $z^m C(z^{-1})$ and $z^m D(z^{-1})$ have all their zeros inside the unit circle. To fulfill these constraints we incorporate a penalty function in (2.30).

Thus, introduce the penalty function

$$Q(\theta) = \frac{1}{\mu}\left[\sum_{i=1}^{m} |z_i(\theta)|^{100} + |p_i(\theta)|^{100}\right] \tag{2.32}$$

where $\mu > 0$, typically 10^4, z_i and p_i are the zeros of $z^m C(z^{-1})$ and $z^m D(z^{-1})$.

Instead of minimizing (2.30), minimize the following function

$$V(\theta) = V_N(\theta) + Q(\theta) \tag{2.33}$$

Let $\hat{\theta}_N$ be the minimizing parameter vector of this criterion. Note that the penalty function (2.32) accept unstable models if needed c.f. section 4, figure 4.6. Remark: Q(θ) can be defined in many ways and (2.32) is just one way of defining it.

Consider the continuous-time model described in section 2.1. Let (2.8) be rewritten as in (2.29) with the following modifications

$$A(\theta) = F(\theta_c)$$

$$v(t,\theta) = v(t,\theta_c) \tag{2.34}$$

$$Ev(t,\theta)v(t,\theta)^T = R_d(\theta_c)$$

$$\theta_c = [c_{r+p-q-m} \ldots c_{r+p}\ d_1 \ldots d_{r+p}\ \lambda_{e_c}^2\ \lambda_w^2]$$

where θ_c is the continuous-time parameter vector .

The notation $\lambda_{e_c}^2$ stands for the variance parameter of the continuous time white noise and λ_w^2 is the variance of the discrete-time white noise. The idea is then the same as for the discrete-time approach and equations (2.29) - (2.30) will apply even for the continuous-time model with the modifications (2.34).

Regarding the penalty function defined in (2.32) we have to do some modifications. In the continuous-time case we want to ensure the poles and zeros of the input model to lie in the left half plane. A suggestion for the continuous-time case is then

$$Q(\theta_c) = \mu[\sum_{\substack{p_i \\ z_i}} (|\mathrm{Re}\ z_i| + \mathrm{Re}\ z_i)^{1.1} + (|\mathrm{Re}\ p_i| + \mathrm{Re}\ p_i)^{1.1}] \qquad (2.35)$$

where $\mu \geq 0$, typically 10^4, z_i and p_i (depending on θ_c) are the zeros of C(s) and D(s) defined in (2.3).

We can finally summarize the continuous-time approach to follow the same pattern as the discrete-time counterpart provided that the modifications (2.34) and (2.35) are taken into account.

It shall be emphasized that the continuous-time approach will render a lot more computations since for every evaluation of $V(\theta_c)$ in (2.33) we have to compute (2.8) and (2.10).

2.5 OPTIMAL FILTERING

When we have found the minimizing parameter vector $\hat{\theta}_N$ to the criterion (2.33), we can put this in to equation (2.31) to obtain an estimate of the input u(t). However, since we are dealing with an off-line algorithm the estimate of u(t) can be improved by using all the data, not only data up to time t-1. The best choice is then to apply optimal fix point smoothing. A time-invariant fix point smoother is given by the following equations, see Anderson and Moore (1979).

$$\hat{x}(t+1|t) = A\hat{x}(t|t-1) + K\tilde{y}(t) \qquad (2.36a)$$

$$\tilde{y}(t) = y(t) - C\hat{x}(t|t-1) \qquad (2.36b)$$

$$\hat{x}(t|t+n) = \hat{x}(t|t+n-1) + K_n^a\ \tilde{y}(t+n) \qquad (2.36c)$$

$$K_n^a = P[A - KC]^{T^n} C^T[CPC^T + R_2]^{-1} \tag{2.36d}$$

$$P = APA^T + R_1 - APC^T(CPC^T + R_2)^{-1}CPA^T \tag{2.36e}$$

$$K = APC^T(CPC^T + R_2)^{-1} \tag{2.36f}$$

Note that equations (2.36a) - (2.36f) depend on the estimated parameter vector $\hat{\theta}_N$ and that R_1, R_2 are the same as in (2.31). The stability properties of (A-KC) for the predictor defined through (2.31) are discussed in Appendix 2.

The time invariant fix point smoother defined by (2.36) does not take the transient effects into account. Hence it is not truly optimal. If we are not interested in estimating the transient behavior of the input u(t), the time invariant smoother (2.36) will save a lot in algorithm simplicity as well as computation time. Further, if {e(t)} and {w(t)} in (2.16) are normal distributed then the estimate (2.36c) will have optimal variance properties.

Let the expected accuracy of the obtained estimate of u (t) be defined by

$$E\tilde{u}(t|t+k)^2 = E[u(t) - \hat{u}(t|t+k)]^2 \tag{2.37}$$

where

$$\hat{u}(t|t+k) = \hat{s}^{(n)}(t) = H\hat{x}(t|t+k) \tag{2.38}$$

and H is defined by (2.8), (2.13) and (2.29).

If the number of lags is taken to k, then (2.37) becomes

$$E\tilde{u}(t|t+k)^2 = H[P - PQ_kP^T]H^T \tag{2.39}$$

where

$$Q_k = \sum_{j=0}^{k} (A-KC)^{T^j} C^T(CPC^T + R_2)^{-1}C(A-KC)^j \tag{2.40}$$

When k tends to infinity (2.37) becomes

$$E\tilde{u}(t|\infty)^2 = E[u(t) - \hat{u}(t|\infty)]^2 = H[P - PQ_\infty P^T]H^T \tag{2.41}$$

and Q_∞ can be computed by solving a Lyapunov equation associated with (2.40). Equation (2.41) gives the accuracy of the optimal time invariant fix point smoother (2.36) for $k = \infty$.

In practice, the gain in smoothing will be achieved in two or three dominant time constants of the Kalman filter (2.31). The number of data available, N, will also be finite. Therefore it is reasonable to stop the estimation procedure when the accuracy (2.39) differ somewhat from the optimal one (2.41). An attempt to achieve this is to use the stop criterion

$$\frac{\sum_{t=1}^{N} [\hat{u}(t|t+k)-\hat{u}(t|t+k-1)]^2}{\sum_{t=1}^{N} [\hat{u}(t|t+k-1)]^2} < \varepsilon \qquad (2.42)$$

where ε can be chosen as e.g. 10^{-4}. The expected accuracy of the estimate $\hat{u}(t|t+k)$ will then be that of (2.39).

3 PRACTICAL ASPECTS

In Section 2 we used a state space formulation for the identification of the input model. In the discrete-time case it will be computationally faster to solve the spectral factorization problem using a polynomial formalism, cf Kucera (1979), instead of solving the Riccati equation (2.31b). See Appendix 3 for details.

The parameter vectors θ and θ_c of (2.28), (2.34) can be reduced to contain c_i, d_i and the ratio of the variances λ_e^2, $\lambda_{e_c}^2$ and λ_w^2 i.e.

$$\theta = [c_1 \ldots c_m \; d_1 \ldots d_m \quad \lambda_e^2/\lambda_w^2]^T \qquad (3.1)$$

$$\theta_c = [c_{r+p-q-m} \ldots c_{r+p} \; d_1 \ldots d_{r+p} \quad \lambda_{e_c}^2/\lambda_w^2]$$

In the following when we talk about the discrete-time ratio λ_e^2/λ_w^2 the same discussion can be applied for the continuous-time case with $\lambda_{e_c}^2/\lambda_w^2$.

It is only the ratio λ_e^2/λ_w^2 that affects the solution of the Riccati equation (2.31b), not the absolute values of λ_e^2 and λ_w^2. Hence the loss function $V(\theta)$ defined by (2.33) will only depend on λ_e^2/λ_w^2. Even the filter properties will depend on λ_e^2/λ_w^2 and it is therefore important to find a correct noise ratio λ_e^2/λ_w^2.

Roughly speaking we can say that, with (2.31) and (2.36) in mind, if λ_e^2/λ_w^2 is taken too large we believe our measurements to be almost "noise free". That means, there will be a large feedback gain K from the output and if we then have some unexpected noise in the measurements, the estimate of u(t) can become quite inaccurate. On the other hand if λ_e^2/λ_w^2 is taken too small i.e. we believe that there is a lot of noise on the measurements, the gain K of the output will become small. If we then have high frequencies in the signal they will be filtered out and the estimated u(t) may be too smooth. In practice it is often better to have a smoother estimate. So maybe if desired, one can incorporate a penalty term accounting for large noise ratios in (2.33).

The discussion above indicates that if the the minimization problem of (2.33) is illconditioned in the direction of λ_e^2/λ_w^2 the estimated u(t) can be bad.

Since we have to use a numerical optimization algorithm in order to find the minimum of (2.33) things can go wrong in the computations. These problems are connected with optimization theory for which a good reference is Gill et al (1981). To avoid troubles, the following rules of thumb for the minimization and model validation can be used. (Newton optimization methods are assumed).

For the minimization:

1. Determine appropriate initial values. Take for instance θ_o = $(0....0 \quad r_o)$ where $r_o = \lambda_e^2/\lambda_w^2$. Any apriori knowledge of λ_w^2 can be used to give a crude approximation of r_o. In the continuous-time case it would be more appropriate to start in a way so that the poles and zeros are situated in the left half plane, for instance in s = -1.

2. The problem can be illconditioned. Investigate the eigenvalues and eigenvectors of the Hessian. If needed, scale the problem or restart with a modified Hessian. Problems of this kind are treated thoroughly in Gill et al (1981).

For the model validation:

3. When finding the minimum of (2.33) have a look att the pole-zero locations. Poles and zeros near each other indicate that the model order is probably chosen too high. If the noise ratio λ_e^2/λ_w^2 is too extreme verify it by restarting the minimization with other initial values.

4. Use standard techniques to determine if the result can be improved by increasing the order of the input model.

In the next section it is shown by numerical examples that identification of the input model will give good results.

4. NUMERICAL ILLUSTRATIONS

The aim of this section is to illustrate numerically the proposed methods for estimating the derivative of a noise corrupted signal. It would of course be possible to suggest many different types of signals as test examples. The following ones have been selected.

1, 2 Continuous-time stationary stochastic processes

3 A sinusoid

4 A polynomial in t

5 Reference data given by Pezzack et al (1977)

First and second order derivatives will be estimated using the three approaches described in section 2.1-2.3. They are

- Continuous time model (CM)
- Discrete time model, constant sampling (DMC)
- Discrete time model, linear sampling (DML)

The signals 1 and 2 are assumed to be described as k times differentiable Gauss-Markov processes with rational spectral densities

$$\Phi_s(\omega) = \frac{B(i\omega)\ B(-i\omega)}{A(i\omega)\ A(-i\omega)} \tag{4.1}$$

where

$$\begin{aligned} A(s) &= s^n + a_1 s^{n-1} + \ldots + a_n \\ B(s) &= b_{k+1}\ s^{n-k-1} + \ldots + b_n \end{aligned} \tag{4.2}$$

are defined through the spectral factorization theorem, cf Åström (1970) and section 2.1.

For signals 1 and 2 CM will certainly be a perfect description, but even DMC and DML will have an accurate structure since we have chosen to describe the input u(t) as an ARMA-process. This can be seen by sampling the continuous-time process. Then we will get a discrete- time filter that coincides with the assumed ARMA input model. This means that it is possible to compute the theoretical values of the discrete-time parameter vector (3.1).

Regarding signals 3 and 4 we can just obtain approximate descriptions of the input model, no matter which one we choose.

Signal 5 contains reference film data given by Pezzack et al (1977) with added white noise as suggested by Lanshammar (1982). The estimated derivative will in this case be compared to reference accelerometer data. The signal to noise ratio for the various signals 1-4 is defined by

$$S/N = \frac{\lambda_s^2}{\lambda_w^2} = \frac{\frac{1}{\tau}\int_0^\tau s(t)^2 \, dt}{\lambda_w^2} \qquad (4.3)$$

where the mean values have been chosen to zero. For signals 1-2 the numerator in (4.3) is replaced by $Es(t)^2$.

The signals to be differentiated, as well as assumptions for the identification are further specified in table 4.1.

Signal	k	T	λ_s^2	λ_w^2	N	*M*	Figure
1. Stoch. process	1	0.2	1	0.01	500	GM(2,1)	4.1
$A(s)=s^2+0.2s+1$						ARMA(2,2)	
$B(s)=0.6325$							
2. Stoch. process							
$A(s)=s^3+2s^2+2s+1$	1,2	0.2	1	0.01	500	GM(3,2)	4.2,4.3
$B(s)=\sqrt{3}$						ARMA(2,2)	
3. Sinusoid							
$s(t)=$	1,2	0.2	1	0.01	500	GM(2,1)	4.4,4.5
$=\sqrt{2}\sin t$						GM(3,2)	
						ARMA(2,2)	
4. Polynomial							
$s(t)=\sqrt{3}t-$	1,2	0.2	1.003	10	500	GM(2,1)	4.6,4.7
$-0.26t^2$						GM(3,2)	
						ARMA(1,1)	
5. Pezzack et al	2	0.0201	-	36.10^{-6}	142	GM(3,2)	4.8
reference						ARMA(2,2)	
data							

<u>Table 4.1</u> Signals to be differentiated

k = the order of derivative

T = sampling interval

λ_s^2 = variance of the signal as defined by (4.3)

λ_w^2 = variance of the measurement noise

N = number of data

M = assumed input model, where ARMA(x,y) is the input model for DMC and DML. The input model for CM is GM(x,y) which stands for Gauss-Markov, A-polynomial order x, B polynomial order y.

Figure = figure(s) where the true and estimated derivative are illustrated.

To describe the derivatives u(t) for DMC and DML a second order ARMA model was used in all cases except for signal 4 where a first order model was assumed to be more accurate. For CM the order of the Gauss-Markov model was chosen so that s(t) would be a stationary process, cf section 2.1.

To evaluate the goodness of the estimated derivatives the following criteria have been used.

$$F1 = \frac{1}{N} \sum_{t=1}^{N} (u(t) - \hat{u}(t))^2 \qquad (4.4a)$$

$$F2 = \frac{\sum_{t=1}^{N} (u(t) - \hat{u}(t))^2}{\sum_{t=1}^{N} u(t)^2} \qquad (4.4b)$$

Simulation results for signal 1-4 are summarized in table 4.2 and figures 4.1-4.7. For signal 5 see figure 4.8.

Signal	Input model DMC	k=1 $\overline{F2}$	k=1 σ_{F2}	k=2 $\overline{F2}$	k=2 σ_{F2}
1	ARMA (2,2)	$4.605\ 10^{-2}$	$0.193\ 10^{-2}$	-	-
2	"	$4.482\ 10^{-2}$	$0.469\ 10^{-2}$	$3.042.10^{-1}$	$0.078.10^{-1}$
3	"	$1.835\ 10^{-2}$	$0.237\ 10^{-2}$	$1.678\ 10^{-2}$	$0.195\ 10^{-2}$
4	ARMA (1,1)	$0.661\ 10^{-2}$	$0.054\ 10^{-2}$	1.338	0.246

Table 4.2 Arithmetic mean $\overline{F2}$ and standard deviation σ_{F2} for first, k=1, and second, k=2, order derivative. Signal to noise ratio = 100. Ten realizations. Constant sampling.

Signal	Input model DML	k=1 $\overline{F2}$	k=1 σ_{F2}	k=2 $\overline{F2}$	k=2 σ_{F2}
1	ARMA(2,2)	$3.195\ 10^{-2}$	$0.129\ 10^{-2}$	-	-
2	"	$2.622\ 10^{-2}$	$0.169\ 10^{-2}$	$2.916\ 10^{-1}$	$0.051\ 10^{-1}$
3	"	$3.266\ 10^{-3}$	$0.532\ 10^{-3}$	$1.879\ 10^{-2}$	$0.560\ 10^{-2}$
4	ARMA(1,1)	$0.728\ 10^{-2}$	$0.137\ 10^{-2}$	0.848	0.384

Table 4.3 Arithmetic mean $\overline{F2}$ and standard deviation σ_{F2} for first, k=1, and second, k=2 order derivative. Signal to noise ratio = 100. Ten realizations. Linear sampling.

Tables 4.2 and 4.3 show for DMC and DML the arithmetic means and standard deviations of F2 for ten realizations of signal 1-4, 500 data points, first and second order derivative. A comparison of table 4.2 and 4.3 tells us mainly that DML is somewhat better than DMC. This was expected since the discrete-time system obtained with DML is a better approximation of the continuous-time system than the one obtained with DMC. Samples of table 4.2 and 4.3 will be shown in figures 4.1-4.7.

The following remarks should be noted for figures 4.1-4.8.

- In all the figures we show true and estimated derivatives. The estimated derivative is obtained with the identified input model. The figures are organized in the following order from the top: DMC, DML, CM.

- Below the figures are also given F2, number of lags = L and the identified noise ratio λ_e^2/λ_w^2 for DMC, DML and CM respectively. The number of lags, L, is defined by (2.42)

- The estimated derivatives are based on 500 data points in figure 4.1-4.7 and 142 data points in figure 4.8.

- The initial values of the estimated derivatives are taken to zero.
- The number of displayed samples are 100 for figures 4.1-4.5, 500 for figures 4.6-4.7 and 142 for figures 4.8.
- The displayed samples show a significant part of the estimated derivative.
- The x-axis displays the time.

For signals 1-2 it is possible to compute the expected optimal accuracy according to (2.41). The accuracy obtained by use of the estimated input model will be compared with the optimal accuracy. Table 4.4 shows the differences when using the realizations depicted in figures 4.1-4.3.

		Estimated input model			True input model	
			F1		F1	EF1
Signal	k	DMC	DML	CM	CM	CM OPT.
1	1	$5.248\ 10^{-2}$	$3.704\ 10^{-2}$	$3.721\ 10^{-2}$	$3.719\ 10^{-2}$	$3.906\ 10^{-2}$
2	1	$1.931\ 10^{-2}$	$1.169\ 10^{-2}$	$1.183\ 10^{-2}$	$4.044\ 10^{-2}$	$1.289\ 10^{-2}$
2	2	$3.346\ 10^{-1}$	$3.259\ 10^{-1}$	$3.175\ 10^{-1}$	$3.043\ 10^{-1}$	$3.013\ 10^{-1}$

Table 4.4. Accuracy with use of estimated input model and true input model. CM OPT. stands for expected optimal accuracy assuming the true continuous model and is computed according to (2.41).

From table 4.4 we see that less good results were obtained for the true input model parameters in two cases. This will probably depend on the realization. We also see that the accuracy F1, obtained with the identified input model parameters approached the expected optimal accuracy very well for DML and CM and quite well for DMC. We can conclude that the identification routine tries to adjust the input model parameters so that it suits the actual realization as well as possible.

Regarding figure 4.1 we see that the estimated derivative follow the true derivative quite well in all cases except for the highest frequencies. DML and CM show however a better tracking then DMC. This is also indicated by F2, see below the figure. Another thing worth noting is the low number of lags, L=10. It says that the major part of information from time t is achieved in 10 future samples.

Figure 4.2 shows the same pattern as figure 4.1 i.e. good tracking for the low frequencies and best behavior for DML and CM. In this case the differences between the estimates are smaller.

It is clear that the difficulties increase when higher order derivatives have to be estimated. Since higher frequencies then occur, it will be more difficult to separate the signal from the noise. This is illustrated in figure 4.3. We see that the true second derivative jumps very much and the estimated derivatives try to follow as good as possible. Further there are not any large difference between the three estimates in this case as indicated by F2.

Regarding figures 4.4 and 4.5 we see that CM give the best tracking in both cases. Both first and second order derivatives are estimated accurately. Note that it seems to be easier to find the frequency than the amplitude. Here it will be in order to make a remark. If we specially choose the system $G(s) = 1$, then this approach can be used to find sinusoids from noisy measurements.

Signal 4 was chosen as a second order polynomial which means that the first order derivative will be a linear function in t. Hence a first order input model was assumed.

As mentioned before the penalty function (2.32), (2.35) accepts unstable models if needed. Signal 4 will be such a case. It seems reasonable to assume that the first order derivative of signal 4 has been generated by an unstable filter, see figure 4.6. We see that DMC and DML follow to the true derivative better than CM.

For the second order derivative displayed in figure 4.7 we obtain better results with CM than DML and DMC. Here we would like a smoother estimate. One way to guarantee that, is to account for large noise ratios in the loss function (2.33) as described in section 3. As we can see below figure 4.7 the noise ratios are already small but perhaps not small enough.

It seems from figure 4.6-4.7 that the initial values θ_o will play a crucial role for the behavior of the estimated derivative, cf discussion in section 3. This means that it sometimes can be fruitful to restart the optimization with other initial values θ_o. One could expect CM to give a better result in figure 4.6 since it gives the best result in figure 4.7.

Finally regarding figure 4.8 the signal is reference angular data with added white noise, having standard deviation σ_w = 0.006 rad, as suggested by Lanshammar (1982). The true derivative is reference accelerometer data. These data as well as the reference angular data are given by Pezzack et al (1977). We see from figure 4.8 that DML and CM give the best result. CM is perhaps better since it has a smoother behavior in the first part of the derivative.

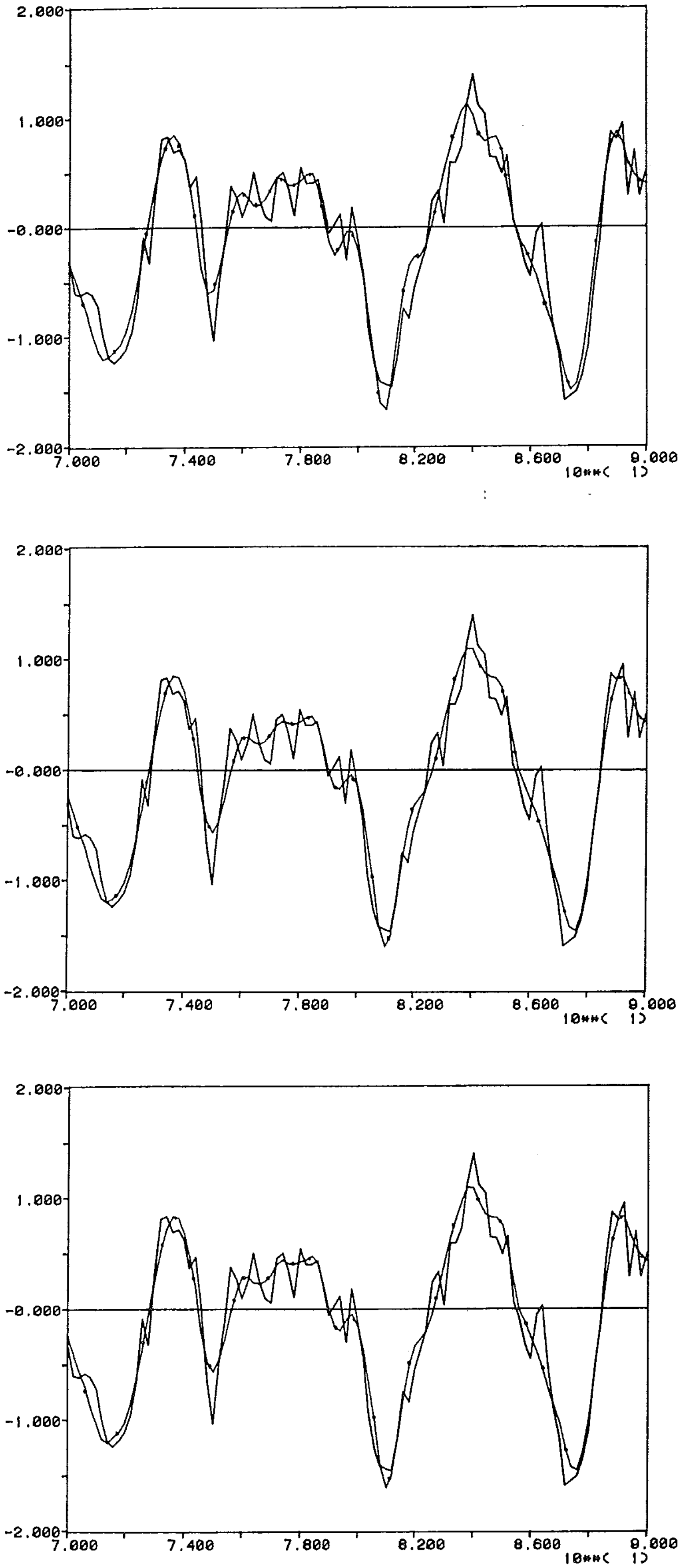


Figure 4.1 Second order stochastic process (signal 1), first order derivative, S/N = 100, input model order = ARMA(2,2), GM(2,1). Solid line = true derivative, line with dots = estimated derivative. F_2 = {4.392 10^{-2}, 3.100 10^{-2}, 3.114 10^{-2}}, λ_e^2/λ_w^2 = {2.97, 2.32, 5.17}, L = {10, 10, 10}.

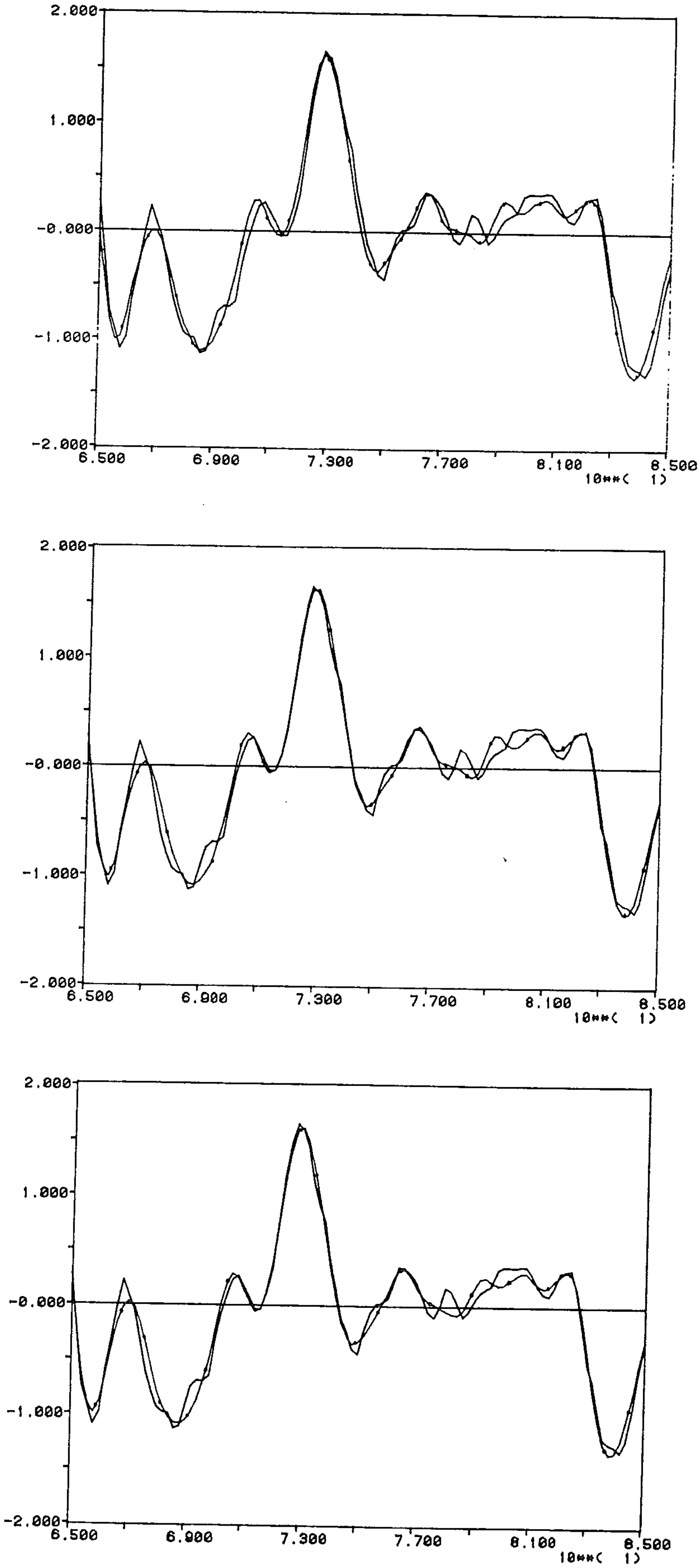
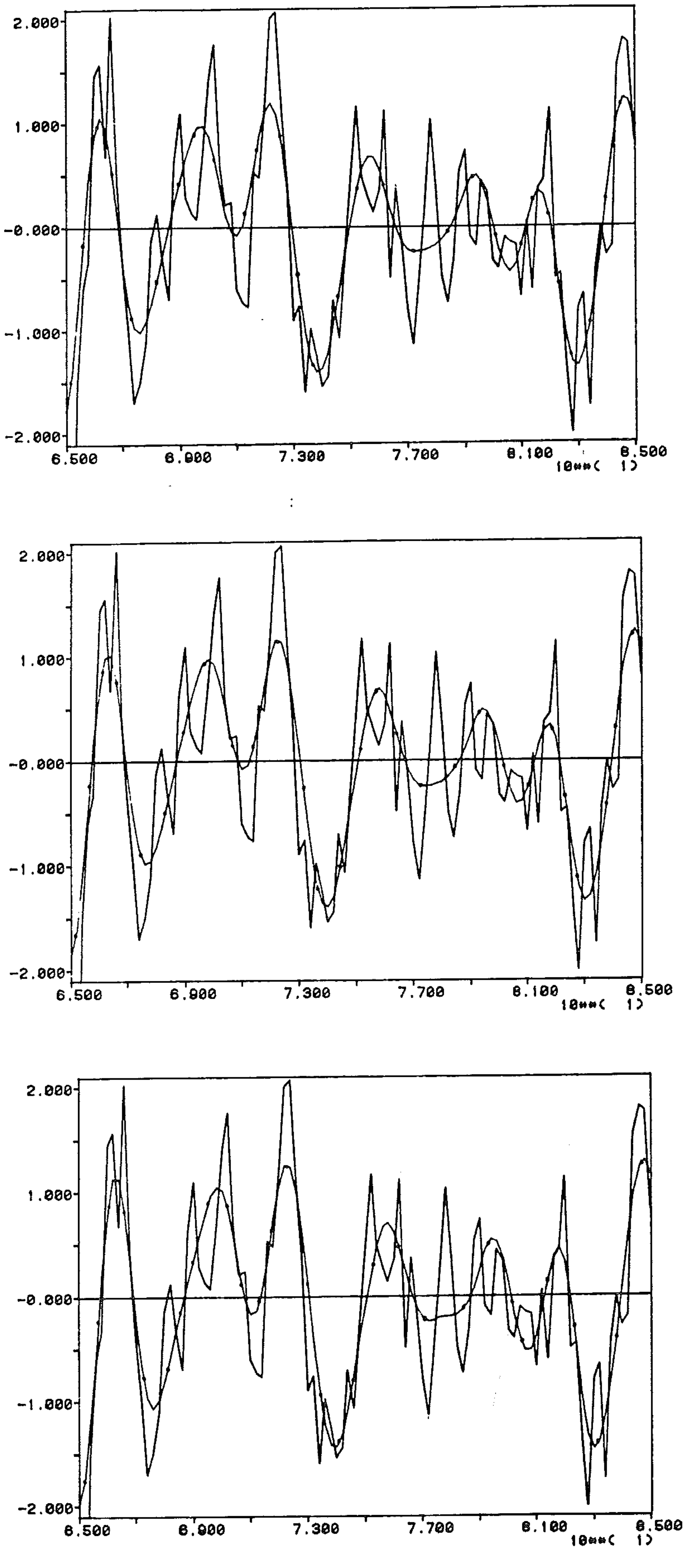


Figure 4.2 Third order stochastic process (signal 2), first order derivative, S/N = 100, input model = ARMA(2,2), GM(3,2). Solid line = true derivative, line with dots = estimated derivative. F2 = {3.830 10^{-2}, 2.320 10^{-2}, 2.346 10^{-2}}, λ_e^2/λ_w^2 = {2.98, 1.16, 10.02}, L = {16, 10, 10}.



<u>Figure 4.3</u> Third order stochastic process (signal 2), second order derivative, S/N = 100, input model = ARMA(2,2), GM(3,2). Solid line = true derivative, dotted line = estimated derivative. F2 = {2.957 10^{-1}, 2.880 10^{-1}, 2.806 10^{-1}}, λ_e^2/λ_w^2 = {2.68, 2.55, 12.75}, L = {14, 14, 19}.

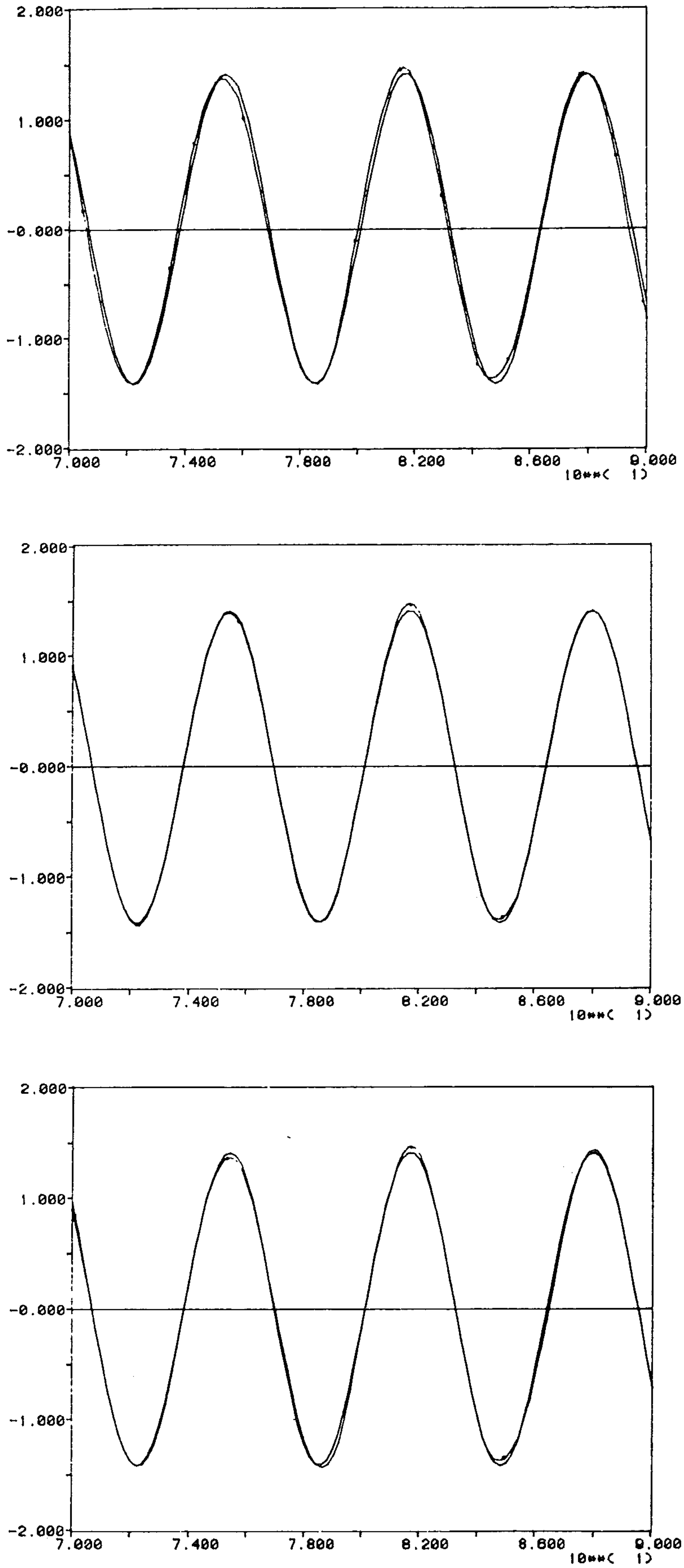


Figure 4.4 Sinusoid (signal 3), first order derivative, S/N = 100, input model = ARMA(2,2), GM(2,1). Solid line = true derivative, line with dots = estimated derivative. F2 = $\{1.518\ 10^{-2},\ 3.959\ 10^{-3},\ 2.896\ 10^{-3}\}$, λ_e^2/λ_w^2 = {0.58, 0.93, 2.99}, L = {10, 9, 15}.

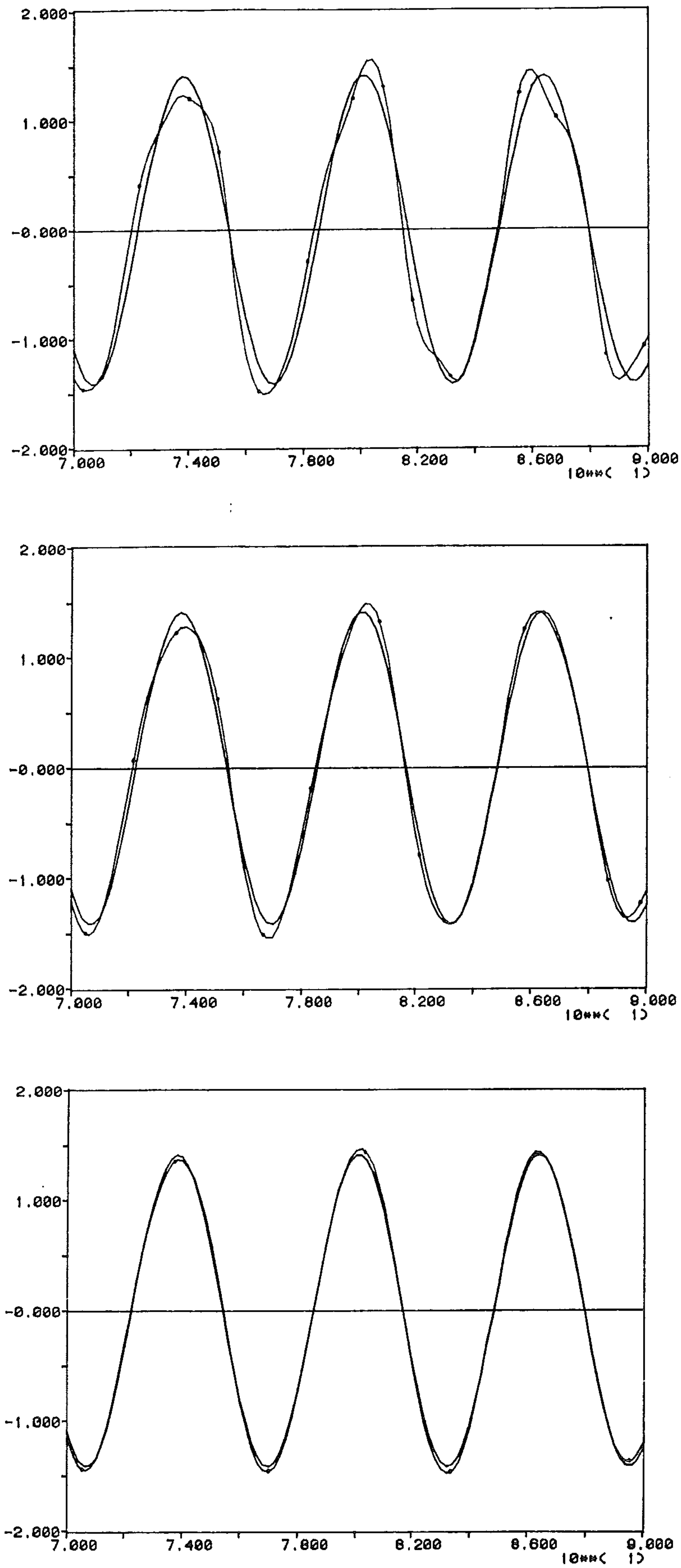


Figure 4.5 Sinusoid (signal 3), second order derivative, input model = ARMA(2,2), GM(3,2). Solid line = true derivative, line with dots = estimated derivative. F2 = {1.415 10^{-2}, 1.419 10^{-2}, 5.707 10^{-3}}, λ_e^2/λ_w^2 = {3.77, 2.13, 2.31}, L = {20, 15, 22}.

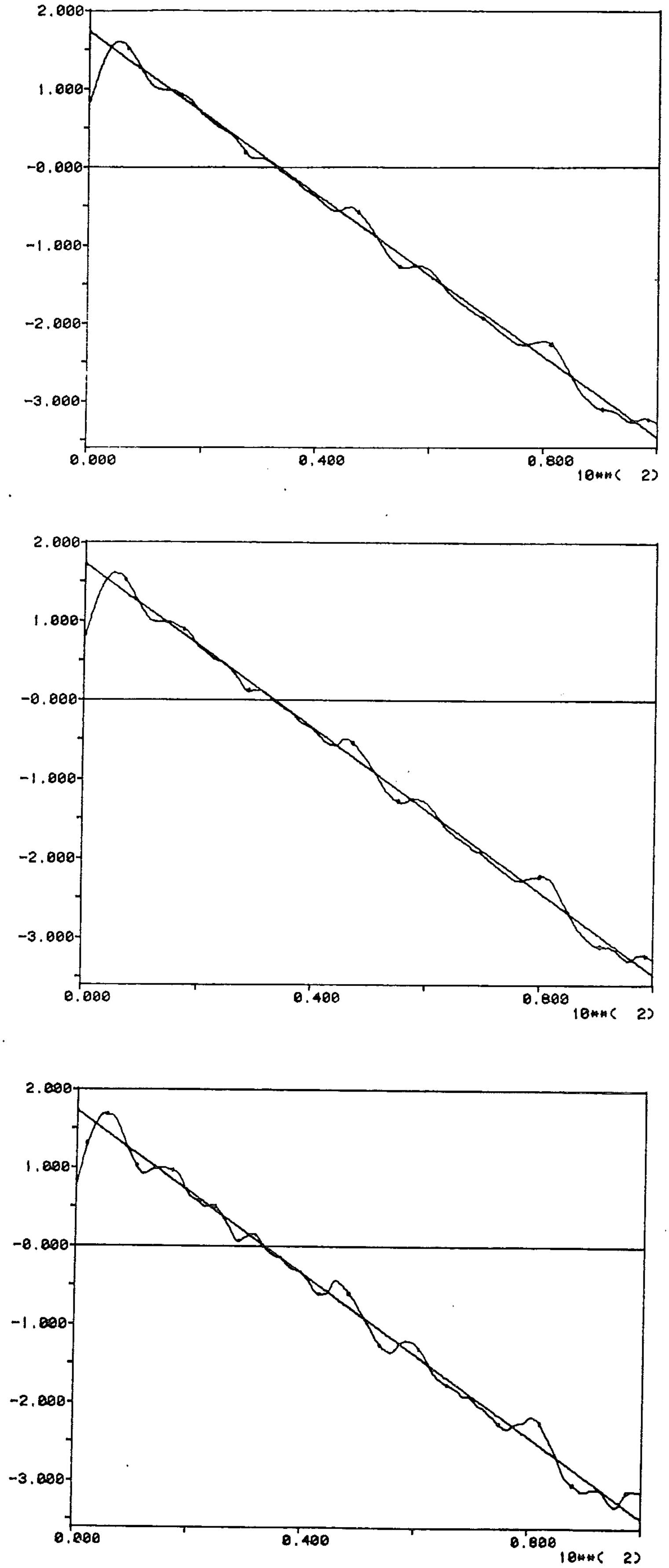


Figure 4.6 Second order polynomial (signal 4), first order derivative, S/N = 100, input model = ARMA(1,1), GM(2,1). Solid line = true derivative, line with dots = estimated derivative. F2 = {0.565 10^{-2}, 0.603 10^{-2}, 0.811 10^{-2}}, λ_e^2/λ_w^2 = {2.63 10^{-4}, 1.00 10^{-2}, 3.39 10^{-4}}, L = {41, 41, 37}.

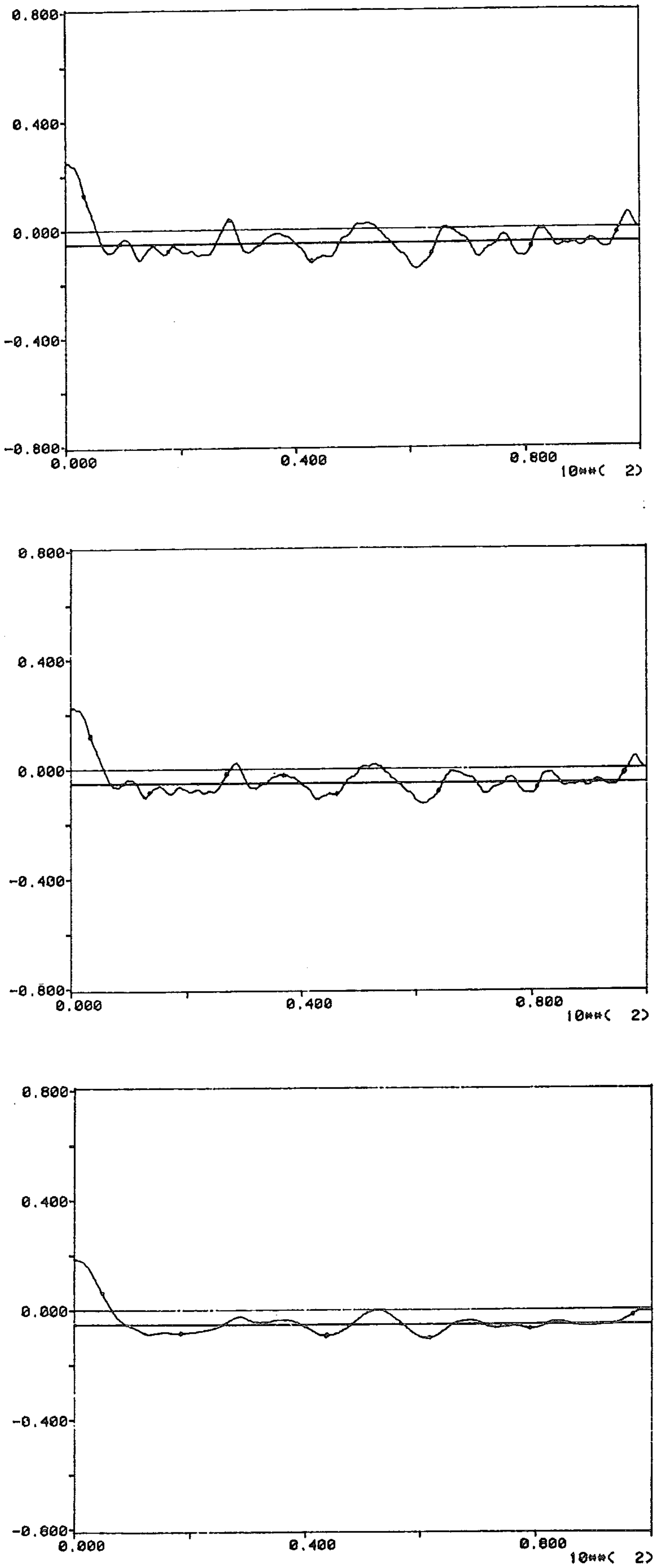


Figure 4.7 Second order polynomial (signal 4), first order derivative, S/N = 100, input model = ARMA(1,1), GM(3,2). Solid line = true derivative, line with dots = estimated derivative. F2 = {1.869, 1.291, 0.939}, λ_e^2/λ_w^2 = {8.32 10^{-3}, 1.72 10^{-2}, 2.44 10^{-3}}, L = {57, 61, 63}.

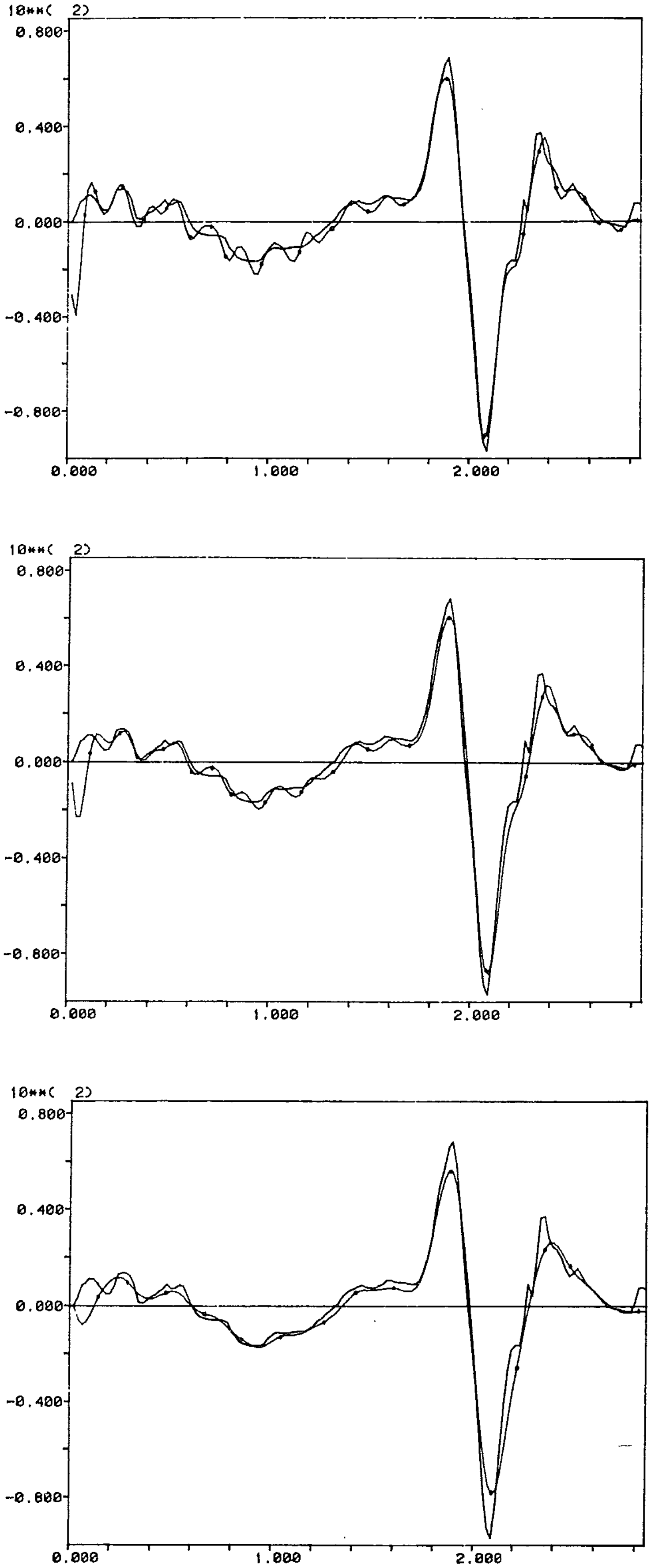


Figure 4.8 Reference accelerometer data given by Pezzack et al. Second order derivative, input model = ARMA(2,2), GM(3,2). Solid line = "true" derivative, line with dots = estimated derivative. F2 = {5.17 10^{-4}, 4.12 10^{-4}, 4.17 10^{-4}}, λ_e^2/λ_w^2 = {2.63 10^5, 1.96 10^5, 1.98 10^5}, L = {13, 10, 21}.

5. SOME COMPARISONS

We have investigated three different ways to obtain good derivative estimates. One is based on a continuous-time description of the signal and the system CM and two are based on discrete-time descriptions, DMC and DML. We have seen from section 4 that DML and CM give very good results. Also DMC will generally give good results.

The difference between DMC and DML is that DML allows a direct term and is obtained by linear sampling, cf sections 2.2, 2.3. The linear sampling gives us a better approximation of the system than the constant sampling.

The continuous-time description CM is of course more accurate than DML but it needs to be sampled every time the optimization routine requires a function value. CM is also tied to the Kalman filter to get the innovation sequence. For DML the faster polynomial approach is applicable, cf Appendix 3. When dealing with higher order derivatives, say > 2, CM usually requires a higher order of the input model than DML due to the fact that u(t) is assumed to be a stationary process. Take as an example signal 4, second order derivative. Then the order of C(s) in (2.2) have to be chosen to 2 and consequently the order of D(s) to 3. The system order is 2 which makes the total order of (2.7) equal to 5. Compare this to DMC and DML for which a first order input model was sufficient. The total order in this case becomes 4.

Another thing that should be noted is the choice of sampling interval. If it is chosen too large aliasing effects will occur. On the other hand, if it is chosen very small the number of data has to be increased so that we cover the same observation time. The signal to noise ratio in (4.3) will otherwise be increased. The best thing is certainly to use a small sampling interval and a great number of data. Then the signals will be approximated much better and the estimate of the derivatives will be improved. Remark: From Åström et al (1984) we know that the zeros of the sampled system will be unstable if the sampling interval becomes "too small". We then suspect to get in trouble with the estimates of u(t) since the input model can become unstable, see (2.16a). However, the penalty functions defined in (2.32) (2.35) will prevent this to happen. If the input model anyhow becomes unstable this will not affect the behavior of the estimate $\hat{u}(t)$ if the system defined by (2.29) is detectable and stabilizable, cf Appendix 2.

We finally conclude that provided the sampling interval is appropriate the most attractive choice of the three approaches is DML.

CONCLUSIONS

We have investigated the problem of estimating the derivative of a noise corrupted signal. The derivative was assumed to be the input of a known linear system i.e. an integrator. The input was modelled as an ARMA or a Gauss-Markov process. In the first phase the parameters of the input model were estimated using a prediction error method. In the second phase, the system was set up into state space form with the input chosen as one of the state variables. Then a time invariant fix point smoother was applied. When formulating the problem in such a way, optimal variance properties of the input estimates are guaranteed, provided that the noise sequences is normal distributed and that we have found an accurate model in phase one.

Derivatives of five different test signals were estimated with three different approaches DMC, DML and CM. The estimated derivatives were compared with the true ones. The results show that the input estimation technique, as formulated in this report, gives very good results. It is also argued that DML is preferable. Due to the good results obtained in section 4 and the simplicity in the choice of user parameters, that is the order of the polynomials in the input model, the proposed methods would be most attractive. In section 4 we also saw that in the discrete-time case an ARMA(2,2) model will often be sufficient.

In the discussed approaches we determined an input model that suited the whole data set in a quadratic sense. We could modify the input model to be time varying in the off-line case and adaptive if the on-line case is considered. Then the input model can be adjusted to follow changes in the dynamics of the signal. How this can be done lies outside the aim of this report and is a topic for current research.

ACKNOWLEDGEMENT

The author is very grateful to Professor Torsten Söderström for valuable guidance and criticism.

REFERENCES

R.S. Anderssen and P. Bloomfield (1974)
A time series approach to numerical differentiation. Technometrics, Vol 16, pp 69-75

B.D.O. Anderson and J.B. Moore (1979)
Optimal Filtering. Prentice-Hall, Englewood Cliffs.

K.J. Åström (1970)
Introduction to Stochastic Control Theory. Academic Press, New York.

K.J. Åström, P. Hagander and J. Sternby (1984)
Zeros of sampled systems. Automatica, Vol. 20, No. 1, pp 31-38.

J.V. Candy and J.E. Zicker (1982)
Deconvolution of noisy transient signals. A Kalman filtering application. IEEE Conference on Decision and Control, Orlando, Florida.

P.E. Gill, W. Murray and M.H. Wright (1981)
Practical Optimization. Academic Press. New York.

L. Gustafsson and H. Lanshammar (1977)
ENOCH - An integrated system for measurement and analysis of human gait. Report UPTEC 7723 R, Institute of Technology, Uppsala University.

H. Hatze (1981)
The use of optimally regularized fourier series for estimating higher-order derivations of noisy biomechanical data. Journal of Biomechanics, Vol. 14, pp. 13-18.

A.J. Jakeman and P. Young (1978)
Systems identifications and estimation for convolution integral equations. In The application and numerical solution of integral equations, R.S. Anderssen et al. (eds.). Sijthoff and Noordhoff.

A.J. Jakeman and P.C. Young (1981)
Recursive filtering and smoothing procedures for the inversion of ill-posed causal problems. Submitted for publication.

V. Kucera (1979)
Discrete Linear Control. Wiley, New York.

H. Kwakernaak and R. Sivan (1972)
Linear Optimal Control Systems. Wiley-Interscience, New York.

H. Lanshammar (1982)
On practical evaluation of differentiation techniques for human gait analysis. Journal of Biomechanics, Vol 15, pp 99-105.

L. Ljung (1976)
On the consistensy of prediction error identification methods. In R.K. Mehra and D.G. Lainiotis, eds: Systems Identification: Advances and Case Studies, Academic Press, New York.

J.C. Pezzack, R.W. Norman and D.A. Winter (1977)
An assessment of derivative determining techniques used for motion analysis. Journal of Biomechanics, Vol 10, pp 377-382.

L.R. Rabiner and K. Steiglitz (1970)
The design of wide band recursive and nonrecursive digital differentiators. IEEE Transactions on Audio and Electroacoustics, Vol. AU-18, No. 2, pp 204-209.

N.R. Sandell, Jr, E. Emre, J.R. Delaney, D.L. Kleinman (1981)
Application of an input estimation technique to determine the acceleration of a maneuvering vehicle. Proceedings IEEE Conference on Decision and Control, San Diego, CA, pp 625-627.

S. Strmcnik and F Bremsak (1979)
Some new transformation algorithms in the identification of continuous-time multivariable systems using discrete identification methods. Preprints 5th IFAC Symposium on Identification and System parameter estimation, Darmstadt, Federal Republic of Germany, pp 397-405.

T. Söderström (1980)
On the design of digital differentiating filters. Report UPTEC 8017 R, Institute of Technology, Uppsala University.

T. Söderström (1982)
Estimation of time derivative from noisy data. Report UPTEC 8290 R, Institute of Technology, Uppsala University.

J.K. Tugnait (1983)
Spectral estimation for noisy signals observed through a linear system. Proceedings IEEE Conference on Decision and Control, San Antonio, TX, pp 1331-1336.

S. Usui and I. Amidror (1982)
Digital low-pass differentiation for biological signal processing. IEEE Transactions on Biomedical Engineering, Vol. BME 29, No. 10, pp 686-693.

G. Wahba (1975)
Smoothing noisy data with spline functions. Numer. Math. Vol 24, pp 383-393.

APPENDIX 1

Proof of Lemma 2.1

The solution of (2.21a) can be described by

$$x(t+T) = e^{AT}x(t) + \int_0^T e^{A(T-\tau)}Bu(t+\tau)d\tau \qquad (A\ 1.1)$$

Assume u(t) to be a linear function between the sampling points i.e.

$$u(t+\tau) = u(t) + \frac{\tau}{T}(u(t+T) - u(t)) \qquad (A\ 1.2)$$

and substitute (A 1.2) in to (A 1.1). Then we obtain

$$x(t+T) = e^{AT}x(t) + \int_0^T e^{A(T-\tau)}B(1 - \frac{\tau}{T})d\tau\ u(t) + \int_0^T e^{A(T-\tau)}B\ \frac{\tau}{T}\ d\tau\ u(t+T)$$

$$= Fx(t) + G_1u(t) + G_2u(t+T) \qquad (A\ 1.3)$$

We have then to compute the following integrals I1 and I2. They are if A is nonsingular

$$I_1 = \int_0^T e^{A(T-\tau)}d\tau = [-A^{-1}e^{A(T-\tau)}]_0^T = A^{-1}(e^{AT}-I) \qquad (A\ 1.4)$$

$$I_2 = \frac{1}{T}\int_0^T e^{A(T-\tau)}\tau d\tau = \frac{1}{T}\ [-A^{-1}e^{A(T-\tau)}\tau]_0^T + \frac{1}{T}\ A^{-1}\int_0^T e^{A(T-\tau)}d\tau \quad (A\ 1.5)$$

$$= \frac{1}{T}\ (-A^{-1}T) + \frac{1}{T}\ A^{-2}(e^{AT}-I)$$

Then we obtain

$$G_1 = [I_1-I_2]B = [A^{-1}e^{AT}-A^{-1} + A^{-1} - \frac{1}{T}\ A^{-2}(e^{AT}-I)]B \qquad (A\ 1.6)$$

$$= \frac{1}{T}\ A^{-2}[e^{AT}(AT-I) + I]B$$

and

$$G_2 = I_2B = \frac{1}{T}\ A^{-2}[e^{AT} - AT - I]B \qquad (A\ 1.7)$$

∎

<u>Proof of Theorem 2.1</u>

Let the system be given by lemma 2.1 i.e. (2.22 a,b). The transfer function from u(t) to s(t) is then given by

$$H(q) = C(qI-F)^{-1}G \qquad \text{(A 1.8)}$$

where

$$G = G_2 q + G_1 \qquad \text{(A 1.9)}$$

Let the continuous-time system $G(s) = 1/s^k$ be represented in observable form i.e.

$$A = \begin{bmatrix} 0 & 1 & & & \\ & \cdot & & 0 & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & 0 & & & 1 \\ & & & & 0 \end{bmatrix} \qquad \text{(A 1.10)}$$

$$B^T = [0 \ldots . 1]$$

$$C = [1 0 \ldots 0]$$

Then $F = e^{AT} = \sum_{j=0}^{k-1} A^j \frac{T^j}{j!}$ since A is nilpotent.

$$F = \begin{bmatrix} 1 & T & T^2/2! & \ldots & T^{k-1}/(k-1)! \\ & \cdot & \cdot & \cdot & \vdots \\ & & \cdot & \cdot & \vdots \\ & & & \cdot & T^2/2! \\ & 0 & & & T \\ & & & & 1 \end{bmatrix} \qquad \text{(A 1.11)}$$

With help of lemma 2.1 we have

$$G_1 = \int_0^T \left(1 - \frac{\tau}{T}\right) \begin{bmatrix} (T-\tau)^{k-1}/(k-1)! \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ T-\tau \\ 1 \end{bmatrix} d\tau = \begin{bmatrix} T^k k/(k+1)! \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ T/2! \end{bmatrix}$$

$$G_2 = \int_0^T \frac{\tau}{T} \begin{bmatrix} (T-\tau)^{k-1}/(k-1)! \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ T-\tau \\ 1 \end{bmatrix} d\tau = \begin{bmatrix} T^k/(k+1)! \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ T/2! \end{bmatrix}$$

which gives

$$G = \begin{bmatrix} T^k/(k+1)!\ (q+k) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ T/2!\ (q+1) \end{bmatrix} \qquad \text{(A 1.12)}$$

We now compute $C(qI-F)^{-1}$.

Put

$$C(qI-F)^{-1} = [a_1 \ldots a_k] \qquad \text{(A 1.13)}$$

Then we have to solve

$$(a_1 \ldots . a_k) \begin{bmatrix} q-1 & -T & \cdot & \cdot & \cdot & \cdot & -T^{k-1}/(k-1)! \\ & \ddots & \ddots & & & & \vdots \\ & & \ddots & \ddots & & & \vdots \\ & & & \ddots & \ddots & & \vdots \\ & 0 & & & \ddots & \ddots & \vdots \\ & & & & & \ddots & -T \\ & & & & & & q-1 \end{bmatrix} = (10 \ldots . 0) \qquad \text{(A 1.14)}$$

We express a_j as

$$a_j = \frac{T^{j-1}}{(j-1)!(q-1)^j} B_j \qquad \text{(A 1.15)}$$

and H(q) will be

$$H(q) = [a_1 \ldots . a_k]G =$$

$$= \frac{T^k}{(q-1)^k} \{ \frac{B_1 (q-1)^{k-1} (q+k)}{(k+1)!\ 0!} + \ldots + \frac{B_k (q+1)}{(k-1)!2!} \} =$$

$$= \frac{T^k}{(k+1)!(q-1)^k} \sum_{j=1}^{k} B_j \binom{k+1}{j-1} (q-1)^{k-j} (q+k+1-j) \qquad \text{(A 1.16)}$$

We now have to determine B_j, $j=1,\ldots,k$

The j:th equation of (A 1.14) is (for $j>1$)

$$a_j(q-1) = a_1 T^{j-1}/(j-1)! + \ldots + a_{j-1} T/1! \qquad \text{(A 1.17a)}$$

We also have

$$a_1(q-1) = 1 \qquad \text{(A 1.17b)}$$

If (A 1.15) is inserted into (A 1.17a) we have

$$B_j = (j-1)! \{ \frac{B_1 (q-1)^{j-2}}{0!(j-1)!} + \frac{B_2 (q-1)^{j-3}}{1!(j-2)!} + \ldots + \frac{B_{j-1}}{(j-2!)\ \ 1!} \} \qquad \text{(A 1.18)}$$

which can be rewritten as

$$B_j = \sum_{p=1}^{j-1} B_p \binom{j-1}{p-1} (q-1)^{j-1-p} \qquad \text{(A 1.19a)}$$

Similarly (A 1.15) and (A 1.17b) give

$$B_1 = 1 \qquad \text{(A 1.19b)}$$

This completes the proof. ∎

APPENDIX 2. STABILITY PROPERTIES OF (A-KC)

Consider the system

$$y(t) = \frac{C(q^{-1},\theta)\ B(q^{-1})}{D(q^{-1},\theta)\ A(q^{-1})} e(t) + w(t) \qquad \text{(A 2.1)}$$

or equivalently rewritten as in (2.29)

$$x(t+1,\theta) = A(\theta)x(t,\theta) + B(\theta)\bar{e}(t)$$

$$y(t) = C(\theta)x(t) + w(t) \qquad \text{(A 2.2)}$$

$$u(t) = H(\theta)x(t)$$

A time invariant optimal predictor is given by

$$\hat{x}(t+1|t;\theta) = A(\theta)\hat{x}(t|t-1;\theta) + K(\theta)(y(t) - C(\theta)\hat{x}(t|t-1;\theta)$$

$$\hat{y}(t|t-1;\theta) = C(\theta)\hat{x}(t|t-1;\theta) \qquad \text{(A 2.3)}$$

Then if the system (A 2.2) is detectable and stabilizable there exists a non-negative definite solution P to the stationary Riccati equation (2.31b), (2,36e) so that $[A(\theta) - K(\theta)C(\theta)]$ is asymptotically stable.

Proof See Kwakernaak - Sivan (1972), pp 535.

If especially the numerator and the denominator in (A 2.1) are coprime then the system (A 2.2) will be completely observable and controllable. This implies that the above conditions are fulfilled.

APPENDIX 3. OBTAINING THE PREDICTION ERRORS BY SPECTRAL FACTORIZATION

Consider the system given by (2.29). From (2.16a) we see that it can be described by

$$y(t) = \frac{P(q^{-1},\theta)}{S(q^{-1},\theta)} e(t) + w(t) \tag{A 3.1}$$

where e(t) and w(t) are defined by (2.16c). The system can also be described by

$$y(t) = \frac{\beta(q^{-1},\theta)}{S(q^{-1},\theta)} \tilde{y}(t) \tag{A 3.2}$$

where $\beta(q^{-1},\theta)$ is defined by the spectral factorization theorem as the asymptotically stable solution of

$$\lambda_e^2 P(e^{i\omega},\theta)P(e^{-i\omega},\theta) + \lambda_w^2 S(e^{i\omega},\theta)S(e^{-i\omega},\theta) = \lambda_{\tilde{y}}^2 \beta(e^{i\omega},\theta)\beta(e^{-i\omega},\theta) \tag{A 3.3}$$

The prediction error required in (2.30) is then obtained by

$$\tilde{y}(t) = \frac{S(q^{-1},\theta)}{\beta(q^{-1},\theta)} y(t) \tag{A 3.4}$$

An algorithm that solves (A 3.3) can be found in Kucera (1979).